CompSci Journal
Double Blind Peer Reviewed, Open Access International Research Journal.
ISSN: 3067-3089 | *Website: https://csjour.com | Volume-1, Issue-6 | December - 2025*

Original Article

Impact of Common Fixed Point Theorems Involving Cubic Terms of d (p, q) in b-Metric Space

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Manuscript ID: CSJ-2025-010605

Volume 1

Issue 6

Pp. *15-22*

December 2025

ISSN: 3067-3089

Submitted: 06 Nov. 2025

Revised: 15 Nov. 2025

Accepted: 05 Dec. 2025

Published: 31 Dec. 2025

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Quick Response Code:

Web. https://csjour.com/

DOI: 10.5281/zenodo.18542731

DOI Link: https://doi.org/10.5281/zenodo.18542731

Abstract

Objective/Aim: To demonstrate the existence and uniqueness of fixed points for self-maps in b-metric spaces, we utilized a generalized ϕ-weak contractive condition that incorporates cubic terms of d(x, y), along with the weak compatibility of two mappings within the context of b-metric spaces. Our findings include the proof of various fixed point theorems for a self-map, as well as common, point theorems for two mappings, accompanied by appropriate examples to substantiate the established results. The innovative aspect of our work lies in proving the existence of fixed points for mappings that meet generalized ϕ-weak contractive conditions with cubic terms of d(x, y) in b-metric spaces, a result that has not been previously established by others. Furthermore, our results serve to extend,discribe and generalize the findings of Dutta and Choudhury, Murthy and Prasad, Hao and Guan, Jain and Kaur, Petru, sel, A. and Petru, sel, J., Peng et al., among others.Ues for other fixed point theorem and improve with work.

Mathematics Subject Classification: *54H25,47H10.*

Keywords: *Fixed Point, b-Metric Space, α−Weak Contraction, Generalized μ−Weak Contraction, Weakly Compatible Mapping, Soft Set, Soft Fixed Point.*

Introduction

Fixed point Theory is one of the most important and useful tool in many branches of Science, Economics, Engineering and Computer Science, etc. The majority of applied mathematics problems reduce to fixed point problems. The famous Banach contraction principle proves the existence and uniqueness of fixed points for a contraction mapping in complete metric space. It has been generalized by many researchers by finding fixed point of single map or common fixed points of two or more mapping in various spaces satisfying various contractive conditions which are either commuting or weakly commuting or compatible or weakly compatible etc. In the development of various metric spaces, one of the significant generalizations of metric space is b- metric space introduced by Bakhtin(1)in 1989. Afterwards, Czerwik(2)proved some fixed point theorems in b-metric spaces in1993.

Theorem1.1: " Let (Z, d) be a complete metric space and S be a φ − contraction mapping on a Z . i. e d(Sp ,Sq)≤ φ { d(p, q)} for all p, q ∈ Z where φ: [0,∞) → [0, ∞) is an upper semi-continuous function from right such that 0 ≤ φ (t)< t for all t > 0 . Then S has a unique fixed point."

In 1997 Alber and Guerre-Delabriere introduced φ − weak contraction and proved the following.

Theorem1.2: Let (Z, d) be a complete metric space and φ − weak contraction mapping on Z. i.e. , d(Sp ,Sq)≤ d(p, q) - φ { d(p, q)} for all p, q ∈ Z, where φ: [0,∞) → [0, ∞) is an upper semi-continuous function from right such that φ (t) > 0 for all t > 0and φ (0) = 0 . Then S has a unique fixed point.

Theorem 1.3 : Let (Z, d) be a complete metric and S be a self map of Z satisfying the following condition:

[1 + rd(p, q)] $d^2(Sp, Sq)$ ≤ r max { $\frac{1}{2}$ [d^2 (p,Sp) d(q,Sq) +d(p,Sp)d^2(q,Sq)],[d(p,Sp)d(p,Sq) d(q,Sp)], d(p, Sq) d(q, Sp) d(q, Sq)}

Where

k(p,q) =

How to cite this article:
Mishra, K. (2025). Impact of Common Fixed Point Theorems Involving Cubic Terms of d (p, q) in b-Metric Space. CompSci Journal, 1(6), 15–22. https://doi.org/10.5281/zenodo.18542731

CompSci Journal
Double Blind Peer Reviewed, Open Access International Research Journal.
ISSN: 3067-3089 *Website: https://csjour.com | Volume-1, Issue-6 | December- 2025*

$\max\left\{d^2(p,q), d(p,Sp)d(q,Sp),\ d(q,Sp)d(p,Sp),\ \frac{1}{2}[d(p,Sp)d(p,Sq) + d(q,Sp)d(q,Sq),]\right\}$

In the present paper, we generalize the above result in the setting of b-metric space and extend our result for a pair of weakly compatible self mappings in b-metric space. We also obtain some suitable examples to justify the proven results. Our results extend and generalize the results of Dutta and Choudhury(13), Murthy and Prasad(12), Hao and Guan(7), Jain and Kaur(8), Petrusel, A. and Petrusel, J.(10), Peng et. al.(11)etc.

Preliminaries:

We now give some known definitions and standard notations which will be needed in the sequel:

Definition2.1

Let a nonempty set Z and let $s \geq 1$ be a given real number. A function $d:Z\times Z\to[0,\infty)$ is said to be b-metric if the following conditions hold good:

(1) $d(g,h) = 0$ if and only if $g = h$.

(2) $d(g,h) = d(h,g)$.

(3) $d(g,h) \leq s[d(g,r)+d(r,h)]$, for all g, h and $r \in Z$.

Then (Z ,d ,s) is called a **b-metric** or metric type space.

Definition 2.2

"The b-metric space (Z, d, s) is called **complete** if every Cauchy sequence in z is convergent in Z."

Definition 2.3

"Let (Z, d, s) be a b-metric space and $S,T:Z\to Z$ be two mappings. The mapping S and T are said to be **weakly compatible** if they commute at their coincident points, i.e., TSp = STp whenever Sp = Tp."

Main Results:

Theorem 3.1.

Let (Z, d, s) be a complete b-metric space and T be a self-map of Z satisfying

$[1 + rd(p,q)]\ d^2(Sp,Sq) \leq \frac{r}{s}\max\ \{ \frac{1}{2}\ [d^2(p,Sp)\ d(q,Sq) + d(p,Sp)\ d^2(q,Sq)\ ,\frac{d^2(q,Sp)}{1+d(q,Sp)}],[d(p,Sp)\ d(p,Sq)\ d(q,Sp) + \frac{d^2(p,Sq)}{1+d(p,Sq)(q,Sp)}],\ d(p,Sq)\ d(q,Sp)\ d(q,Sq)\ \} + m(p,q) - \varphi\{m(p,q)\}$ (3.1.1)

Where

$k(p,q) = \max\left\{d^2(p,q), d(p,Sp)d(q,Sp)\frac{d^2(q,Sp)}{1+d(q,Sp)},\ d(q,Sp)d(p,Sp),\ \frac{1}{2s}[d(p,Sp)d(p,Sq),\left(\frac{d^2(q,Sp)}{1+d(q,Sp)}\right) + d(q,Sp)d(q,Sq),\left(\frac{d^2(p,Sq)}{1+d(p,Sp)(q,Sp)}\right)]\right\}$

$r \geq 0$ is a real number and $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$ and $\varphi(t) > 0$ for each $t > 0$. Then, T has a unique fixed point in Z.

Proof: Let p_0 be an arbitrary point in Z.We define a sequence $\{p_n\}$in Z $.p_{n+1} = T\,p_n$, $n \geq 0$. (3.1.2)

If $p_n = p_{n+1}$, for some n, then, T has trivially a fixed point. Therefore we assume that $p_n \neq p_{n+1}$ for all $n \in N$ .For convenience

$d(p_n,p_{n+1}) = d_n$ (3.1.3)

First,we show that $\{d_n\}$ is a monotonically decreasing sequence, i.e., $d_{n+1} \leq d_n$.

Case1: If n is odd. Take $n = 2m-1$. We prove that $d_2 \leq d_{2m-1}$.

Now taking $p = p_{2m-1}, q = p_{2m}$ in equality (3.1.1)

we have $[1+ r\ d(p_{2m-1}, p_{2m})]\ d^2(p_{2m-1},Sp_{2m}) \leq \frac{r}{s}\ \max\ \frac{1}{2}\ [d^2(p_{2m-1}, S\,p_{2m-1})\ d(p_{2m}, Sp_{2m}) + d(p_{2m-1},Sp_{2m-1})d^2(p_{2m},Sp_{2m}),\frac{d^2(p_{2m},Sp_{2m-1}}{1+d(p_{2m},Sp_{2m-1})}],[d(p_{2m-1},Sp_{2m-1})d(p_{2m-1},Sp_{2m})\ d(p_{2m},Sp_{2m-1}) + \frac{d^2(p_{2m},Sp_{2m-1})}{1+d(p_{2m},Sp_{2m-1})d(p_{2m-1},S\,p_{2m}}],d(p_{2m-1},Sp_{2m})\ d(p_{2m},Sp_{2m})\ d(p_{2m}, Sp_{2m})\ \} + m(p_{2m-1}, p_{2m})\ -$

$\varphi\ [m(p_{2m-1}, p_{2m})]$

Where

$$K(p_{2m-1}, p_{2m}) = \max\left\{\begin{matrix} d^2(p_{2m-1}, p_{2m}), d(p_{2m-1}, Sp_{2m-1})d(p_{2m}, Sp_{2m-1}) \\ \left(\frac{d^2(p_{2m}, Sp_{2m-1})}{1+d(p_{2m}, Sp_{2m-1})}\right), d(p_{2m}, Sp_{2m-1})d(p_{2m-1}, Sp_{2m-1}), \\ \frac{r}{2s}[d(p_{2m-1}, Sp_{2m-1})d(p_{2m-1}, Sp_{2m})\left(\frac{d^2(p_{2m}, Sp_{2m})}{1+d(p_{2m}, Sp_{2m})}\right) + \\ d(p_{2m}, Sp_{2m-1})d(p_{2m}, Sp_{2m}), \left(\frac{d^2(p_{2m-1}, Sp_{2m})}{1+d(p_{2m-1}, Sp_{2m-1})d(p_{2m-1}, Sp_{2m})}\right)] \end{matrix}\right\}$$

Now using Inequality (3.1.2)

$[1+ r\, d(p_{2m-1}, p_{2m})]\, d^2(p_{2m}, p_{2m+1}) \le \frac{r}{s} \max\{\frac{1}{2}[d^2(p_{2m-1}, p_{2m})\, d(p_{2m}, p_{2m+1}) + d(p_{2m-1}, p_{2m})\, d^2(p_{2m}, p_{2m+1}), \frac{d^2(p_{2m}, p_{2m})}{1+d(p_{2m}, p_{2m})}], [d(p_{2m-1}, p_{2m})\, d(p_{2m-1}, p_{2m+1})\, d(p_{2m}, p_{2m}) + \frac{d^2(p_{2m}, p_{2m})}{1+d(p_{2m}, p_{2m})d(p_{2m-1}, p_{2m})}], d(p_{2m-1}, p_{2m+1})\, (p_{2m}, p_{2m})\, d(p_{2m}, p_{2m+1})\} + m(p_{2m-1}, p_{2m}) - \varphi[m(p_{2m-1}, p_{2m})]$

$$k(p_{2m-1}, p_{2m}) = \max\left\{\begin{matrix} d^2(p_{2m+1}, p_{2m}), d(p_{2m-1}, p_{2m})d(p_{2m}, p_{2m})\left(\frac{d^2(p_{2m}, p_{2m})}{1+d(p_{2m}, p_{2m})}\right) \\ , d(p_{2m}, p_{2m})\, d(p_{2m-1}, p_{2m}), \\ \frac{r}{2s}[d(p_{2m-1}, p_{2m})\, d(p_{2m-1}, p2m+1), \left(\frac{d^2(p_{2m}, p_{2m+1})}{1+d(p_{2m}, p_{2m})}\right) + d(p_{2m}, p_{2m+1}), \\ \left(\frac{d^2(p_{2m-1}, p_{2m+1})}{1+d(p_{2m-1}, p_{2m})\, d(p_{2m-1}, p_{2m+1})}\right)] \end{matrix}\right\}$$

$[1+ r\, d_{2m-1}]\, d^2_{2m} \le \frac{r}{s} \max\{\frac{1}{2}[d^2_{2m-1}\, d_{2m} + d_{2m-1}d^2_{2m}, 0, 0, 0\} + m(d_{2m-1}, d_{2m}) - \varphi(d_{2m-1}, d_{2m})]$ (3.1.4)

Where

$k(p_{2m-1}, p_{2m},) = \max\left\{d^2_{2m-1}, d^2_{2m-1}\, d_{2m}\, d_{2m}\ 0, \frac{r}{2s}[d^2_{2m-1}\, d(p_{2m-1}, p_{2m+1}), 0, 0]\right\}$

$k(p_{2m-1}, p_{2m}) = \max\left\{d^2_{2m-1}, d^2_{2m-1}\, d_{2m}\, d_{2m}\ 0, \frac{r}{2s}[d^2_{2m-1}\, r(d^2_{2m-1} + d^2_{2m}), 0, 0]\right\}$

[By triangular inequality of b-metric space]

If $d^2\, 2m-1 < d_{2m}$

Then

$m(p2m-1, p2m) < \max\{d^2_{2m}, 0, 0, d^2_{2m}\} = d^2_{2m}$

So Inequality (3.1.4) becomes

$d^2_{2m} + r\, d^2_{2m-1}\, d^2_{2m} < \frac{r}{s}\max\{\frac{1}{2}[d^2_{2m-1}\, d^2_{2m} + d^2_{2m-1}\, d^2_{2m}], 0, 0\} + d^2_{2m} - \varphi(d^2_{2m})$.

$\le r\, d^2_{2m-1}\, d^2_{2m} + d^2_{2m} - (d^2_{2m})$.

Which gives $\varphi(d^2_{2m}) < 0$, a contraction .

Thus $d^2_{2m} \le d^2_{2m-1}$

Case 2 : If n is even, taking n = 2m and $p = p_{2m}$

$q = p_{2m+1}$ in Inequality (1) and proceeding as above we can easily obtain that $d^2_{2m+1} \le d^2_{2m}$

Therefore ,$\{d_n\}$ is a monotonically decreasing sequence of non-negative real numbers and convergent.

Let $\log_{n\to\infty} d_n = k$ for some $k \ge 0$ (3.1.5)

We assume that k = 0. Using Inequality (3.1.1), we have

$[1+ r\, d(p_n, p_{n+1})]\, d^2(Sp_n, Sp_{n+1}) \le \frac{r}{s} \max \frac{1}{2}[d^2(p_n, Sp_n)\, d(p_{n+1}, S\,p_{n+1}) + d(p_n, Sp_n)\, d^2(p_{n+1}, Sp_{n+1}), \frac{d^2(p_{n+1}, Sp_n)}{1+d(p_{n+1}, Sp_n)}], [d(p_n, Sp_n)\, d(p_n, S\,p_{n+1})d(p_{n+1}, Sp_n) + \frac{d^2(p_{n+1}, Sp_n)}{1+d(p_{n+1}, Sp_n)d(p_n, S\,p_{n+1})}], d(p_n, Sp_{n+1})\, d(p_{n+1}, Sp_n)\, d(p_{n+1}, S\,p_{n+1})\} + m(p_n, p_{n+1}) - \varphi[m(p_n, p_{n+1})]$

$[1+ r\, d(p_n, p_{n+1})]\, d^2(p_{n+1}, p_{n+2}) \le \frac{r}{s} \max \frac{1}{2}[d^2(p_n, p_{n+1})\, d(p_{n+1}, p_{n+2}) + d(p_n, p_{n+1})\, d^2(p_{n+1}, p_{n+2}), \frac{d^2(p_{n+1}, p_{n+1})}{1+d(p_{n+1}, p_{n+1})}], [d(p_n, p_{n+1})\, d(p_n, p_{n+2})\, d(p_{n+1}, p_{n+1}) + \frac{d^2(p_{n+1}, p_{n+1})}{1+d(p_{n+1}, p_{n+1})d(p_n, p_{n+2})}], d(p_n, p_{n+2})\, d(p_{n+1}, p_{n+1})\, d(p_{n+1}, p_{n+2})\} + m(p_n, p_{n+1}) - \varphi[m(p_n, p_{n+1})]$

This implies that

$(1+ r\, d_n)\, d^2_{n+1} \le \frac{r}{s}\max\{\frac{1}{2}[d^2_n\, d_{n+1} + d_n d^2_{n+1}, 0], 0, 0\} + m(p_n, p_{n+1}) - \varphi[m(p_n, p_{n+1})]$ (3.1.6)

Where

$$k(p_n, p_{n+1}) = \max\left\{\begin{array}{c} d^2(p_n, p_{n+1}),\ d(p_n, Sp_n)d(p_{n+1}, Sp_n) \\ \left(\frac{d^2(p_{n+1}, Sp_n)}{1+d(p_{n+1}, Sp_n)}\right), d(p_{n+1}, Sp_n)d(p_n, Sp_n), \\ \frac{r}{2s}\left[d(p_n, Sp_n)d(p_n, Sp_{n+1}),\ \left(\frac{d^2(p_{n+1}, Sp_n)}{1+d(p_{n+1}, Sp_n)}\right)+\right. \\ \left. d(p_{n+1}, Sp_n)d(p_{n+1}, Sp_{n+1}),\ \left(\frac{d^2(p_n, Sp_{n+1})}{1+d(p_n, Sp_n)d(p_{n+1}, Sp_n)}\right)\right] \end{array}\right\}$$

$$\max\left\{\begin{array}{c} d^2(p_n, p_{n+1}), d(p_n, p_{n+1})d(p_{n+1}, p_{n+1}) \\ \left(\frac{d^2(p_{n+1}, p_{n+1})}{1+d(p_{n+1}, p_{n+1})}\right), d(p_{n+1}, p_{n+1})d(p_n, p_{n+1}), \\ \frac{r}{2s}\left[d(p_n, p_{n+1})d(p_n, p_{n+2}),\ \left(\frac{d^2(p_{n+1}, p_{n+1})}{1+d(p_{n+1}, p_{n+1})}\right)\right. \\ \left. +d(p_{n+1}, p_{n+1})d(p_{n+1}, p_{n+2}),\ \left(\frac{d^2(p_n, p_{n+2})}{1+d(p_n, p_{n+1})d(p_{n+1}, p_{n+1})}\right)\right] \end{array}\right\}$$

Now, using triangle inequality of b- metric space, taking limit as $n \to \infty$ and using Inequality (3.1.5) we have

$$\leq \max\left\{d_n^2,\ \frac{r}{2s}\left[d_n\, d(p_n, p_{n+2})\ 0, 0,\right\}\right.$$

$$\log_{n\to\infty} k(p_n, p_{n+1}) \leq \max\left\{k^2,\ \frac{1}{2s}\left[ks\, d(k+k)\ 0, 0,\right\}\right. = k^2 \qquad (3.1.7)$$

Now taking limit as $n \to \infty$ in inequality (3.1.6) and using Inequality (3.1.7) we have

$(1 + rk)\, k^2 \leq \frac{r}{s} k^3 + k^2 - \varphi k^2 \leq r\, k^3 + k^2 - \varphi(k^2)$

Which gives $\varphi(k^2) \leq 0$, which implies that $k = 0$.

Therefore,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(p_n, p_{n+1}) = 0 \qquad (3.1.8)$$

Now , we show that $\{p_n\}$ is a cauchy sequence in Z. Let m, n ∈ N,m > n , then $d(p_n, p_m) \leq s\{d(p_n, p_{n+1}) + d(p_{n+1}, p_m)\} \leq s\, d(p_n, p_{n+1}) + s^2\{d(p_{n+1}, p_{n+2}) + d(p_{n+2}, p_m)\}$

$\leq s\, d(p_n, p_{n+1}) + s^2 d(p_{n+1}, p_{n+2}) + s^3 d(p_{n+2}, p_{n+3}) + \ldots\ldots\ldots\ldots$

$= s\, d_n + s^2 d_{n+1} + s^3 d_{n+2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Taking limit n→ ∞ and using Inequality (3.1.8) we get $\lim_{n\to\infty} d(p_n, p_m) = 0$

Hence $\{p_n\}$ is a cauchy sequence in Z. But Z is a complete b- matric space and therefore there exists some p∈ Z

Such that $\lim_{n\to\infty} p_n = p \qquad (3.1.9)$

Now ,we prove that p is a fixed point of S.

Replacing q by p_n in inequality (3.1.1),we obtain

$[1 + rd(p, p_n)]\, d^2(Sp, Sp_n) \leq \frac{r}{s} \max\ \{\frac{1}{2}\ [d^2(p,Sp)\, d(q,p_{n+1}) + d(p,Sp)\, d^2(p_n,p_{n+1}), \frac{d^2(p_n,Sp)}{1+d(p_n,Sp)}],\ \ [d(p,Sp)\, d(p,p_{n+1})\, d(p_n,Sp) + \frac{d^2(p,p_{n+1})}{1+d(p,p_{n+1})(p_n,Sp)}], d(p,p_{n+1})\, d(p_n,Sp)\, d(p_n,p_{n+1})\} + m(p, p_n) \quad - \varphi\{m(p, p_n)\}$

$[1 + rd(p, p_n)]\, d^2(Sp, p_{n+1}) \leq r \max\ \{\frac{1}{2}\ [d^2(p,Sp)\, d(q,Sp_n) + d(p,Sp)\, d^2(p_n,Sp_n), \frac{d^2(p_n,Sp)}{1+d(p_n,Sp)}],\ [d(p,Sp)\, d(p,Sp_n)\, d(p_n,Sp) + \frac{d^2(p,Sp_n)}{1+d(p,Sp_n)(p_n,Sp)}], d(p,Sp_n)\, d(p_n,Sp)\, d(p_n,Sp_n)\} + m(p,p_n) \quad - \varphi\{m(p, p_n)\}$

(3.1.10)

Where

$$k(p, p_n) = \max\left\{\begin{array}{c} d^2(p, p_n), d(p, Sp)d(p_n, Sp)\frac{d^2(p_n,Sp)}{1+d(p_n,Sp)}\ , \\ d(p_n, Sp)d(p, Sp),\ \frac{1}{2s}\left[d(p, Sp)d(p, Sp_n), \left(\frac{d^2(p_n,Sp)}{1+d(p_n,Sp)}\right)+\right. \\ \left. d(p_n, Sp)d(p_n, Sp_n), \left(\frac{d^2(p,Sp_n)}{1+d(p,Sp)(p_n,Sp)}\right)\right] \end{array}\right\}$$

$$k(p, p_n) = \max\left\{\begin{array}{c} d^2(p, p_n), d(p, Sp)d(p_n, Sp)\frac{d^2(p_n,Sp)}{1+d(p_n,Sp)}\ , d(p_n, Sp)d(p, Sp), \\ \frac{1}{2s}\left[d(p, Sp)d(p, p_{n+1}), \left(\frac{d^2(p_n,Sp)}{1+d(p_n,Sp)}\right)+\ d(p_n, Sp)d(p_n, Sp_n)\right. \\ \left., \left(\frac{d^2(p,p_{n+1})}{1+d(p,Sp)(p_n,Sp)}\right)\right] \end{array}\right\}$$

Now taking limit as n→ ∞ and using Inequality (3.1.8) and Inequality (3.1.9) we get

$$\lim_{n\to\infty} k(p, p_n) = \max\left\{0,0,0\ \ \frac{1}{2s}\ [0,0 + 0,0]\right\} = 0 \qquad (3.1.11)$$

Now taking limit as n→ ∞ and using Inequality (3.1.10) and Inequality (3.1.11) we have

$[1+0]\, d^2(Sp,p) \le \frac{r}{s}\max\{\frac{1}{2}[0+0,0],[0+0],0\}+0 - \varphi(0) = 0$

This implies that $d^2(Sp,p) \le 0$. Thus $d^2(Sp,p)=0$ and consequently, d(Sp,p) = 0
Thus Sp = p

Uniqueness: Let p and q be two fixed points of S.Then by using Sp = p and Sq = q ,inequality(1)
Reduces to

$[1+rd(p,q)]\, d^2(p,q) \le \frac{r}{s}\max\{\frac{1}{2}[d^2(p,p)\, d(q,q)+d(p,p)\, d^2(q,q), \frac{d^2(q,p)}{1+d(q,p)}],[d(p,p)\, d(p,q)\, d(q,p)+\frac{d^2(p,q)}{1+d(p,q)(q,p)}], d(p,q)\, d(q,p)\, d(q,q)\} + m(p,q) - \varphi\{m(p,q)\}$ (3.1.12)

Where

$k(p,q)=\max\left\{d^2(p,q), d(p,p)d(q,p)\frac{d^2(q,p)}{1+d(q,p)}, d(q,p)d(p,p), \frac{1}{2s}[d(p,p)d(p,q),\left(\frac{d^2(q,p)}{1+d(q,p)}\right) + d(q,p)d(q,q),\left(\frac{d^2(p,q)}{1+d(p,p)(q,p)}\right)]\right\}$

$k(p,q) = \max\{d^2(p,q), 0, 0, d^2(p,q), 0\}$

$= d^2(p,q)$

So Inequality (3.1.12) becomes

$[1+r\, d(p,q)]\, d^2(p,q) \le \frac{r}{s}\max\{0,0,0\} + d^2(p,q) - \varphi\{d^2(p,q)\}$

$rd^3(p,q) \le -\varphi\{d^2(p,q)$

$rd^3(p,q) + \varphi d^2(p,q) \le 0,$

Which is only possible when $d^2(p,q) = 0$.
This implies that p = q .

Hence S has a unique fixed point.
Now we furnish an example that shows the existence of a unique fixed point for a self map in complete b-metric space Z(Z is not a metric space as shown in example 2.3)

Corollary 3.3
Let (Z, d, s) be a complete b- metric space and S be a self –map of Z satisfying

$d^2(Sp,Sq) \le m(p,q) - \varphi(m(p,q))$,

where $M(p,q) = \max\{d^2(p,q), d(p,Sp)\, d(q,Sq), d(p,Sq)d(q,Sp), \frac{1}{2s}[d(p,Sq)d(p,Sq)+d(q,Sp)d(q,Sq)]\}$

R ≥ 0 is a real number and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function with $\varphi(t)=0$ if and only if t = 0 and $\varphi(t) > 0$ for each t > 0. Then S has a unique fixed points in Z.

Now we extend theorem 3.1 by finding a common fixed points result for two self maps using generalized φ – weak cotractive condition with cubic term involving r in the setting of b-metric space.

Theorem 3.2 : Let (Z, d, s) be a complete b- metric space and L, M self maps of Z such that M(Z) ⊂ L(Z) and the pair {L,M} is weakly compatible. Also,

$[1+rd(Lp,Lq)]\, d^2(Mp,Mq) \le \frac{r}{s}\max\{\frac{1}{2}[d^2(Lp,Lp)\, d(Lq,Mq)+d(Lp,Mp)\, d^2(Lq,Mq), \frac{d^2(Lq,Mp)}{1+d(Lq,Mp)}],[d(Lp,Mp)\, d(Lp,Mq)\, d(Lq,Mp)+\frac{d^2(Lp,Mq)}{1+d(Lp,Mq)(Lq,Mp)}], d(Lp,Mq)\, d(Lq,Mp)\, d(Lq,Mq)\} + m(Lp,Lq) - \varphi\{m(Lp,Lq)\}$ (3.2.1)

Where

$$k(Lp,Lq)=\max\left\{\begin{array}{c} d^2(Lp,Lq), d(Lp,Mp)d(Lq,Mp)\frac{d^2(Lq,Mp)}{1+d(Lq,Mp)}, \\ d(Lq,Mp)d(Lp,Mp), \frac{1}{2s}[d(Lp,Mp)d(Lp,Mq),\left(\frac{d^2(Lq,Mp)}{1+d(Lq,Mp)}\right)+ \\ d(Lq,Mp)d(Lq,Mq),\left(\frac{d^2(Lp,Mq)}{1+d(Lp,Mp)(Lq,Mp)}\right)] \end{array}\right\},$$

r ≥ 0 is areal number and and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function with $\varphi(t)=0$ if and only if t = 0 and $\varphi(t) > 0$ for each t > 0. Then L and M have a unique fixed points in Z.

Proof : Let p_0 be an arbitrary point in Z. Since M(Z) ⊂ L(Z) .Therefore we can find some p_1 such that $Mp_0 = Lp_1 = q_1$.

We define a sequence $\{q_n\}$ in Z by $q_n = Mp_{n-1} = L\, p_n$, n ≥ 0 . (3.2.2)

For convenience, we write $d(q_n, q_{n+1}) = d_n$ (3.2.3)

First we show that $\{q_n\}$ is a monotonically decreasing sequence, i.e. $d_{n+1} \le d_n$

Case 1: If n is odd. Take n = 2z-1 .We prove that $d_{2z} \le d_{2z-1}$

Now taking $p = p_{2z-1}$, $q = q_{2z}$ in Inequality(3.2.1),we have

$[1+rd(Lp_{2z-1},Lq_{2z})]\, d^2(Mp_{2z-1},Mq_{2z}) \le \frac{r}{s}\max\{\frac{1}{2}[d^2(Lp_{2z-1},Lp_{2z-1})\, d(Lq_{2z},Mq_{2z}) + d(Lp_{2z-1},Mp_{2z-1})\, d^2(Lq_{2z},Mq_{2z}, \frac{d^2(Lq_{2z},Mp_{2z-1})}{1+d(Lq_{2z},Mp_{2z-1})}],[d(Lp_{2z-1},Mp_{2z-1})\, d(Lp_{2z-1},Mq_{2z})\, d(Lq_{2z},Mp_{2z-1}) + \frac{d^2(Lp_{2z-1},Mq_{2z})}{1+d(Lp_{2z-1},Mq_{2z})(Lq_{2z},Mp_{2z-1})}], d(Lp_{2z-1},Mq_{2z})$

$d(Lq_{2z},Mp_{2z-1})\, d(Lq_{2z},Mq_{2z})\ \} + m\,(Lp_{2z-1},Lq_{2z}) - \varphi\{m(Lp_{2z-1}, L\,q_{2z})\}$

Where

$k(Lp_{2z-1},Lq_{2z})=$

$$\max\left\{\begin{matrix} d^2(Lp_{2z-1},Lq_{2z}), d(Lp_{2z-1},Mp_{2z-1})d(Lq_{2z},Mp_{2z-1}) \\ \frac{d^2(Lq_{2z},Mp_{2z-1})}{1+d(Lq_{2z},Mp_{2z-1})}\ ,\ d(Lq_{2z},Mp_{2z-1})d(Lp_{2z-1},Mp_{2z-1}), \\ \frac{1}{2s}\ [d(Lp_{2z-1},Mp_{2z-1}) \\ d(Lp_{2z-1},Mq_{2z}),\left(\frac{d^2(Lq_{2z},Mp_{2z-1})}{1+d(Lq_{2z},Mp_{2z-1})}\right) + d(Lq_{2z},Mp_{2z-1}) \\ d(Lq_{2z},Mq_{2z}),\left(\frac{d^2(Lp_{2z-1},Mq_{2z})}{1+d(Lp_{2z-1},Mp_{2z-1})(Lq_{2z},Mp_{2z-1})}\right)] \end{matrix}\right\}$$

Now using Inequality (3.2.2) ,we get

$[1 + rd(p_{2z-1},q_{2z})]\ d^2(p_{2z},q_{2z+1}) \le \frac{r}{s}\max\ \{\frac{1}{2}\ [d^2(p_{2z-1},p_{2z-1})\ d(q_{2z},q_{2z+1}) + d(p_{2z-1},p_{2z})\ d^2(p_{2z},q_{2z+1}), \frac{d^2(q_{2z},p_{2z})}{1+d(q_{2z},p_{2z})}],[\ d(p_{2z-1},p_{2z})\ d\ (p_{2z-1},q_{2z+1})\ d(q_{2z},p_{2z}) + \frac{d^2(p_{2z-1},q_{2z+1})}{1+d(p_{2z-1},q_{2z+1})(q_{2z},p_{2z})}],\ d(p_{2z-1},q_{2z+1})\ d(q_{2z},p_{2z})\ d(q_{2z},q_{2z+1})\ \} + m\ (p_{2z-1},q_{2z}) - \varphi\{m(p_{2z-1}, q_{2z})\}$

Where

$$k(Lp_{2z-1},Lq_{2z})= \max\left\{\begin{matrix} d^2(p_{2z-1},q_{2z}), d(p_{2z-1},p_{2z})d(q_{2z},p_{2z})\frac{d^2(q_{2z},p_{2z})}{1+d(q_{2z},Mp_{2z})}\ , \\ d(q_{2z},p_{2z})d(p_{2z-1},p_{2z})\ ,\ \frac{1}{2s}\,[d(p_{2z-1},p_{2z})d(p_{2z-1},q_{2z+1}), \\ \left(\frac{d^2(q_{2z},p_{2z})}{1+d(q_{2z},p_{2z})}\right) + d(q_{2z},p_{2z})d(q_{2z},q_{2z+1}),\left(\frac{d^2(p_{2z-1},q_{2z+1})}{1+d(p_{2z-1},p_{2z})(q_{2z},p_{2z})}\right)] \end{matrix}\right\}$$

Then

$[1 + rd_{2z-1}]d_{2z}^2 \le \frac{r}{s}\max\ \{\frac{1}{2}\ [0 + d_{2z-1}\ \ d_{2z}^2,\ \ 0],[\ 0+0\ ],0\ \} + m\ (p_{2z-1},q_{2z}) - \varphi\{m(p_{2z-1}, q_{2z})\}$ (3.2.4)

Where

$k(Lp_{2z-1},Lq_{2z}) = \max\left\{d_{2z-1}^2, 0\ ,\ 0,\ \frac{1}{2s}\ [d_{2z-1}\, s(d_{2z-1} + d_{2z}\ ),0,0]\right\}$

[By triangle Inequality of b- metric space]

If $d_{2z-1} < d_{2z}$

Then

$M(p_{2z-1}, d_{2z}) < \max\ \{d_{2z}^2, d_{2z}^2\ , 0\ d_{2z}^2\} = d_{2z}^2$

So Inequality(3.2.3) becomes

$d_{2z}^2 + rd_{2z-1}d_{2z}^2 \le \frac{r}{s}\max\ \{\frac{1}{2}\ [d_{2z-1}\ \ d_{2z}^2,\ \ 0,0,0\ \} - \varphi\{d_{2z}^2\}$

$\le r\ rd_{2z-1}d_{2z}^2 + d_{2z}^2\ - \varphi\{d_{2z}^2\}$

$[d_{2z-1}^2 < d_{2z}^2$,gives $- \varphi\{d_{2z}^2\} < - \varphi\{d_{2z-1}^2\}]$

Which gives $d_{2z} \le d_{2z-1}$

Case 2: If n is a even, taking n = 2z and putting $p = p_{2z}, q = d_{2z+1}$ in Inequality (3.1.1) and proceeding as above we can easily obtain that $d_{2z+1} \le d_{2z}$

Therefore $\{d_z\}$ Is a monotonically decreasing sequence of nonnegative real number. Now proceding as in 3.1 we can easily prove that

$\lim_{n\to\infty} d_z = \lim_{n\to\infty} d(q_z, q_{z+1}) = 0$ and $\{q_z\}$ is a Cauchy sequence in Z.

But Z is a complete b- metric space and therefore exists some $q \in Z$ such that

$\lim_{n\to\infty} q_z = q$ (3.2.5)

Since $q \in M(Z) \subset L(Z)$, therefore,there exist a point $v \in Z$ such that $q = Lv$. Now we prove that $Mv = q$. For this replacing p by v and q by p_z in inequality (3.2.1), we obtain

$+ rd(Lv, Lp_z)]\ d^2(Mv, Mp_z) \le \frac{r}{s}\max\ \{\frac{1}{2}\ [d^2(Lv,Lv)\ d(Lp_z,Mp_z) + d(Lv,Mv)\ d^2(Lp_z,Mp_z), \frac{d^2(Lp_z,Mv)}{1+d(Lp_z,Mv)}],[\ d(Lv, Mv)\ d(Lv, Mp_z)\ d(Lp_z,Mv) + \frac{d^2(Lv,Mp_z)}{1+d(Lv,Mp_z)(Lp_z,Mv)}],\ d(Lv,Mp_z)\ d(Lp_z,Mv)\ d(Lp_z,Mp_z)\} + m(Lv,Lp_z) - \varphi\{m(Lv, L\,p_z)\}$ (3.2.6)

Where

$k(Lv,Lp_z)=$

$$\max\left\{\begin{array}{c} d^2(Lv, Lp_z), d(Lv, Mv)d(Lp_z, Mv)\frac{d^2(Lp_z,Mv)}{1+d(Lp_z,Mv)}, d(Lp_z, Mv)d(Lv, Mv), \\ \frac{1}{2s}[d(Lv, Mv)d(Lv, Mp_z), \left(\frac{d^2(Lp_z,Mv)}{1+d(Lp_z,Mv)}\right) + d(Lp_z, Mv)d(Lp_z, Mp_z), \\ \left(\frac{d^2(Lv,Mp_z)}{1+d(Lv,Mv)(Lp_z,Mv)}\right)] \end{array}\right\}$$

$[1 + rd(q,q_z)]\ d^2(Mv, q_{z+1}) \le \frac{r}{s}\max\ \{\frac{1}{2}\ [d^2(q,q)\ d(q_z,q_{z+1}) + d(q,Mv)\ d^2(q_z,q_{z+1}), \frac{d^2(q_z,Mv)}{1+d(q_z,Mv)}], [d(q,Mv)\ d(q,q_{z+1})\ d(q_z,Mv) + \frac{d^2(q,q_{z+1})}{1+d(q,q_{z+1})(q_z,Mv)}], d(q,q_{z+1})\ d(q_z,Mv)\ d(q_z,q_{z+1})\} + m(q,q_z) - \varphi\{m(q, q_z)\}$

Where

$$k(q,q_z) = \max\left\{\begin{array}{c} d^2(q, q_z), d(q, Mv)d(q_z, Mv)\frac{d^2(q_z,Mv)}{1+d(q_z,Mv)}, \\ d(q_z, Mv)d(q, Mv), \frac{1}{2s}[d(q, Mv)d(q, q_{z+1}), \\ \left(\frac{d^2(q_z,Mv)}{1+d(q_z,Mv)}\right) + d(q_z, Mv)d(q_z, q_{z+1}), \left(\frac{d^2(q,q_{z+1})}{1+d(q,Mv)(q_z,Mv)}\right)] \end{array}\right\}$$

Now taking limit as $n \to \infty$ and using Inequality (3.2.5), we get

$d^2(Mv, q) \le \frac{r}{s}\max\ \{0,0,0\} + m(q,q_z)$

$- \varphi\{m(q, q_z)\}$

where

$K(q, q_z) = \max\{0,0,0,0,0\} = 0$

Therefore, $K(q, q_z) \le 0$, which gives q = Mv

Hence Lv = q =Mv . Thus v is coincidence point of L and M . Now since {L,M}is weakly compatible, therefore LMv = MLv

Which gives Lq = Mq.

Now we claim that q is common fixed point of L and M. For this we replace p by v in Inequality(3.2.1)

$[1 + rd(Lv, Lq)]\ d^2(Mv, Mq) \le \frac{r}{s}\max\ \{\frac{1}{2}\ [d^2(Lv,Lv)\ d(Lq,Mq) + d(Lv,Mv)\ d^2(Lq,Mq), \frac{d^2(Lq,Mv)}{1+d(Lq,Mv)}], [d(Lv,Mv)\ d(Lv, Mq)\ d(Lq, Mv) + \frac{d^2(Lv,Mq)}{1+d(Lv,Mq)(Lq,Mv)}], d(Lv,Mq)\ d(Lq,Mv)\ d(Lq,Mq)\} + m(Lv,Lq) - \varphi\{m(Lv, Lq)\}$

Where

$$k(Lv,Lq) = \max\left\{\begin{array}{c} d^2(Lv, Lq), d(Lv, Mv)d(Lq, Mv)\frac{d^2(Lq,Mv)}{1+d(Lq,Mv)}, \\ d(Lq, Mv)d(Lv, Mv), \frac{1}{2s}[d(Lv, Mv)d(Lv, Mq), \\ \left(\frac{d^2(Lq,Mv)}{1+d(Lq,Mv)}\right) + d(Lq, Mv)d(Lq, Mq), \left(\frac{d^2(Lv,Mq)}{1+d(Lv,Mp)(Lq,Mv)}\right)] \end{array}\right\}$$

Since Lv = q = Mv and Mq = Lq ,Therefore

$[1 + rd(q, Lq)]\ d^2(q, Lq) \le \frac{r}{s}\max\ \{\frac{1}{2}\ [d^2(q,q)\ d(Lq, Lq) + d(q, q)\ d^2(Lq, Lq), \frac{d^2(Lq,q)}{1+d(Lq,q)}], [d(q,q)\ d(q,Lq)\ d(Lq,q) + \frac{d^2(q,Lq)}{1+d(q,Lq)(Lq,q)}], d(q, Lq)\ d(Lq, q)\ d(Lq, Lq)\} + m(q,Lq) - \varphi\{m(q, Lq)\}$

Where

$k(q, Lq) = \max\{d^2(q, Lq), d(q, q)d(Lq, q)\frac{d^2(Lq,q)}{1+d(Lq,q)},$

$d(Lq, q)d(q, q), \frac{1}{2s}[d(q, q)d(q, Lq), \left(\frac{d^2(Lq,q)}{1+d(Lq,q)}\right) + d(Lq, q)d(Lq, Lq), \left(\frac{d^2(q,Lq)}{1+d(q,Lq)(Lq,q)}\right)]\}$

$k(q,Lq) = \max\{d^2(q, Lq), 0, 0, \frac{1}{2s}[0,0 + 0,0]\} = d^2(q, Lq)$

Therefore,

$d^2(q, Lq) + r\ d^3(q, Lq) \le \frac{r}{s}.0 + d^2(q, Lq)$

$- \varphi\{d^2(q, Lq)\}$.

This implies that

$rd^3(q, Lq) + \varphi\{d^2(q, Lq)\} \le 0$, which gives d(q,Lq) = 0 and hence q =Lq = Mq .

Thus q is common fixed point of L and M.

Uniqueness: Let c be a another common fixed point of L and M .Then using Inequality (3.2.1) , we have

$[1 + rd(Lc, Lq)]\ d^2(Mc, Mq) \le \frac{r}{s}\max\ \{\frac{1}{2}\ [d^2(Lc,Lc)\ d(Lq,Mq) + d(Lc,Mp)\ d^2(Lq,Mq), \frac{d^2(Lq,Mc)}{1+d(Lq,Mc)}], [d(Lc,Mc)\ d(Lc,Mq)\ d(Lq,Mc) + \frac{d^2(Lc,Mq)}{1+d(Lc,Mq)(Lq,Mc)}], d(Lc,Mq)\ d(Lq,Mc)\ d(Lq,Mq)\} + m(Lc,Lq) - \varphi\{m(Lc, Lq)\}$ (3.2.7)

Where

k(Lc,Lq)= $\max\left\{d^2(Lc, Lq), d(Lc, Mc)d(Lq, Mc)\frac{d^2(Lq,Mc)}{1+d(Lq,Mc)}, d(Lq, Mc)d(Lc, Mc), \frac{1}{2s}[d(Lc, Mc)d(Lc, Mq), \left(\frac{d^2(Lq,Mc)}{1+d(Lq,Mc)}\right) + d(Lq, Mc)d(Lq, Mq), \left(\frac{d^2(Lc,Mq)}{1+d(Lc,Mp)(Lq,Mc)}\right)]\right\}$

$[1 + rd(c,q)]\, d^2(c, q) \leq \frac{r}{s}\max\ \{\frac{1}{2}[d^2(c, c)\, d(q, q) + d(c, p)\, d^2(q, q), \frac{d^2(q,c)}{1+d(Lq,Mc)}], [d(c, c)\, d(L c, q)\, d(q, c) + \frac{d^2(c,q)}{1+d(c,q)(q,c)}], d(c, q)\, d(q, c)\, d(q, q)\} + m(c, q) - \varphi\{m(c, q)\}$ (3.2.8)

$$k(c, q) = \max\left\{\begin{matrix} d^2(c, q), d(c, c)d(q, c)\frac{d^2(q,c)}{1+d(q,c)}, d(q, c)d(c, c), \\ \frac{1}{2s}[d(c, c)d(c, q), \left(\frac{d^2(q,c)}{1+d(q,c)}\right) + d(q, c)d(q, q), \left(\frac{d^2(c,q)}{1+d(c,c)(q,c)}\right)] \end{matrix}\right\}$$

$= d^2(c, q)$

Solving inequality (18),we have c = q.

Consequently ,q is a unique common fixed point of L and M.

Now ,we given example in support of our theorem.

Conclusion:

In the present paper, firstly ,we have established a fixed point theorem using a generalized φ weak contractive condition involving cubic terms of d(p,q) in the setting of b metric space and then we have generalized the results by finding a unique common fixed point of two self mappings. The results proved in this paper generalized and extend various known existing results. Further, some suitable examples are also given which show the generality of our proven results over the existing ones.

Acknowledgment

The author expresses sincere gratitude to the reviewers and editors for their valuable comments and constructive suggestions, which significantly improved the quality and clarity of this manuscript. The author is also thankful to colleagues and members of the Department for their academic support, insightful discussions, and encouragement during the course of this research work. The support provided by the institution is gratefully acknowledged.

Financial support and sponsorship

Nil.

Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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